

# DOUBLY PERIODIC SELF-TRANSLATING SURFACES FOR THE MEAN CURVATURE FLOW

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**ABSTRACT.** We construct new examples of self-translating surfaces for the mean curvature flow by desingularizing a periodic configuration with infinitely many grim reaper cylinders. These surfaces show that self-translating solutions do not necessarily have polynomial volume growth rate.

## 1. INTRODUCTION

A surface  $M \subset \mathbf{R}^3$  is said to be a *self-translating solution* or *soliton* if it satisfies

$$(1) \quad H - \vec{e}_y \cdot \nu = 0,$$

where  $H$  is the mean curvature,  $\nu$  is the unit normal vector such that  $\mathbf{H} = H\nu$  and  $\mathbf{H}$  is the mean curvature vector. Without loss of generality, we have chosen the velocity of the translation to be  $\vec{e}_z$ , the third coordinate vector. As the name indicates, a surface  $M$  is self-translating if and only if the  $t$ -time slice of a mean curvature flow starting at  $M$  is  $M_t = M + t\vec{e}_y$ .

The cylinder over the grim reaper given by

$$(2) \quad \tilde{\Gamma} = \{(x, y, z) \in (-\pi/2, \pi/2) \times \mathbf{R}^2 \mid y = -\ln(\cos x)\}$$

is an example of self-translating surface in  $\mathbf{R}^3$ . Let us denote by  $\tilde{\Gamma}_n$  the grim reaper cylinder  $\tilde{\Gamma}$  shifted by  $(\tilde{b}_n, \tilde{c}_n, 0)$ , for real numbers  $\tilde{b}_n$  and  $\tilde{c}_n$ . With a slight abuse of language, we will also call  $\tilde{\Gamma}$  and its translated apparitions grim reapers. Other examples of solitons include rotationally symmetric surfaces [1], a non-graphical genus one rotationally symmetric soliton [2], and examples desingularizing the intersection of a finite family of grim reapers in general position [5]. In higher dimensions, the existence of convex non  $k$ -rotationally symmetric self-translating surfaces in  $\mathbf{R}^n$ ,  $n \geq k$ , are given by Wang [6].

In this paper, we show the existence of a family of doubly periodic self-translating surfaces. The period in the  $\vec{e}_x$  direction can be taken to be any real number, except  $\pm \frac{\pi}{q}$ ,  $q \in \mathbf{N}$ .

**Theorem 1.** *Given a vector  $\vec{a} = (a_x, a_y, 0) \in \mathbf{R}^3$  with  $a_x \neq 0$  and  $a_x \neq \pm \frac{\pi}{q}$ ,  $q \in \mathbf{N}$ , there exists a one parameter family of surfaces  $\{\tilde{M}\}_{\bar{\tau} \in (0, \delta_{\vec{a}})}$ , with  $\delta_{\vec{a}} > 0$  depending only on  $\vec{a}$  such that  $\tilde{M}_{\bar{\tau}}$  is a surface satisfying (1) that is invariant under the translation by  $\vec{a}$  and periodic of period  $2\pi\bar{\tau}$  in the  $z$ -direction.*

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The surfaces from Theorem 1 show that in general, self-translating surfaces do not have polynomial area growth. We contrast this with the recent result from [3] that provides polynomial volume growth for self-shrinkers.

Theorem 1 follows from an extension of the result from [5]. Indeed, we show below that given a periodic configuration  $G = \bigcup_{j \in \mathbf{Z}} \bigcup_{n=1}^{N_\Gamma} (\tilde{\Gamma}_n + j\vec{a})$  in general position, it is possible to desingularize  $G$  to obtain self-translating surfaces. One proves Theorem 1 by choosing an adequate family  $\bigcup_{n=1}^{N_\Gamma} \tilde{\Gamma}_n$  depending on  $\vec{a}$ . Let us assume from now on that  $\vec{a} \cdot \vec{e}_z = 0$ .

**Definition 2.** A periodic family of grim reapers  $G = \bigcup_{j \in \mathbf{Z}} \bigcup_{n=1}^{N_\Gamma} (\tilde{\Gamma}_n + j\vec{a})$  is said to be in general position if it satisfies the following requirements:

- (i) there exists a constant  $\varepsilon > 0$  such that the distance between any two asymptotic planes to  $G$  is at least  $\varepsilon$ ,
- (ii) no three grim reapers intersect on the same line.

Let us denote by  $\delta$  the minimum of the distance between two intersection points of  $G \cap \{z = 0\}$  and by  $\delta_\Gamma$  the minimum measure of the angles formed by intersecting grim reapers and of the angles the grim reapers form with  $\vec{e}_y$  at the intersection points. Note that both  $\delta$  and  $\delta_\Gamma$  are positive because there are finitely many intersections within a period.

We denote by  $N_I$  the number of intersection lines within a period of  $G$  (i.e. between the planes  $x = c$  and  $x = c + \vec{a} \cdot \vec{e}_x$ , where  $c$  is taken generically);  $N_I$  is different from the number of intersections of  $\bigcup_{n=1}^{N_\Gamma} (\tilde{\Gamma}_n + j\vec{a})$ . We associate a positive integer  $m_k$  to each intersection line, which allows us to take different scales at different intersections, making the construction as general as possible.

**Theorem 3.** Given a finite family of grim reapers  $G = \bigcup_{j \in \mathbf{Z}} \bigcup_{n=1}^{N_\Gamma} (\tilde{\Gamma}_n + j\vec{a})$  in general position, there is a one parameter family of surfaces  $\{\tilde{\mathcal{M}}_{\bar{\tau}}\}_{\bar{\tau} \in (0, \delta_{\bar{\tau}})}$ , with  $\delta_{\bar{\tau}}$  depending only on  $N_I$ ,  $\max_k(m_k)$ ,  $\delta$ , and  $\delta_\Gamma$  satisfying the following properties:

- $\tilde{\mathcal{M}}_{\bar{\tau}}$  is a complete embedded surface satisfying (1).
- $\tilde{\mathcal{M}}_{\bar{\tau}}$  is singly periodic of period  $2\pi\bar{\tau}$  in the  $z$ -direction.
- $\tilde{\mathcal{M}}_{\bar{\tau}}$  is invariant under the translation  $(x, y, z) \mapsto (x + \vec{a} \cdot \vec{e}_x, y + \vec{a} \cdot \vec{e}_y, z)$ .
- If  $U$  is a neighborhood in  $\mathbf{R}^2$  such that  $U \times \mathbf{R}$  contains no intersection line, then  $\tilde{\mathcal{M}}_{\bar{\tau}} \cap (U \times \mathbf{R})$  converges uniformly in  $C^r$  norm, for any  $r < \infty$ , to  $(\bigcup_{j \in \mathbf{Z}} \bigcup_{n=1}^{N_\Gamma} (\tilde{\Gamma}_n + j\vec{a})) \cap (U \times \mathbf{R})$  as  $\bar{\tau} \rightarrow 0$ .
- If  $T_{kj}$  is the translation parallel to the  $xy$ -plane that moves the  $k$ th intersection line of the  $j$ th period to the  $z$ -axis, then  $\bar{\tau}^{-1}T_{kj}(\tilde{\mathcal{M}}_{\bar{\tau}})$  converges uniformly in  $C^r$ , for any  $r < \infty$ , on any compact set of  $\mathbf{R}^3$  to a Scherk surface with  $m_k$  periods between  $z = 0$  and  $z = 2\pi$  as  $\bar{\tau} \rightarrow 0$ .

The proof is similar to the one from [5] where we desingularized a finite family of grim reapers in general position. Many of the steps from the previous article (construction of desingularizing surfaces and initial approximate solutions, study of the associated linear operator, and application of fixed point theorem) are valid here provided one has enough flexibility to change the tangent vectors at each intersection slightly while keeping the initial configuration embedded. The flexibility is needed to deal with the small eigenvalues of the linear operator associated to  $H - \vec{e}_y \cdot \nu$ . In [5], we started by fixing the left sides of the grim reapers and propagating the errors caused by the flexibility to the right. The family of grim reapers

was finite, so the accumulation of errors was bounded. In the present proof, we start by fixing well chosen grim reaper edges and let the free grim reaper ends absorb the movement from the dislocations at the intersections.

This article is to be read as an extension of [5] and we rely on the reader's familiarity with either [5] or [4]. The rest of the article is structured as follows. In the next section, we prove Theorem 1 assuming Theorem 3. In Section 3, we show that our periodic configuration has enough flexibility. In Section 4, we outline the main steps of the proof of Theorem 3.

**Acknowledgment.** The present paper is a result of a discussion with Sigurd Anagnostou on whether the mean curvature  $H$  on  $M \setminus B_R$  tends to 0 as  $R \rightarrow \infty$  for a self-translating surface  $M$ . The author would like to thank him for many discussions and suggesting this counterexample.

## 2. PROOF OF THEOREM 1 GIVEN THEOREM 3

We show that for a vector  $\vec{a}$  as in Theorem 1, we can find a periodic family of grim reapers in general position invariant under translation by  $\vec{a}$ . To simplify the proof, we assume without loss of generality that  $a_x > 0$ .

**Case 1:**  $\pi > a_x > \pi/2$ . This is the simplest case, where we just take  $G = \bigcup_{j \in \mathbf{Z}} \tilde{\Gamma} + j\vec{a}$ . The  $j$ th grim reaper  $\tilde{\Gamma} + j\vec{a}$  intersects only its neighbors  $\tilde{\Gamma} + (j \pm 1)\vec{a}$  and the distance between two asymptotic planes is bounded below by  $\min(\pi - a_x, 2a_x - \pi)$ .

**Case 2:**  $a_x > \pi$ . Let  $K$  be a positive integer such that  $K\pi > a_x > K\pi/2$  (for example,  $K := \lfloor \frac{a_x}{\pi} \rfloor + 1$ ). We can fall back on the first case by considering  $\bigcup_{j \in \mathbf{Z}} \left( \bigcup_{i=0}^{K-1} \tilde{\Gamma} + \frac{i+Kj}{K} \vec{a} \right)$ ; however the configuration is really invariant under translation by  $\vec{a}/K$ . To kill all periods smaller than  $\vec{a}$ , we take  $\{\mathbf{d}_i\}_{i=0}^{K-1}$  a set of distinct vectors in  $\mathbf{R}^3$  for which  $\mathbf{d}_0 = \mathbf{0}$ ,  $\mathbf{d}_i \cdot \vec{e}_z = 0$ , and  $|\mathbf{d}_i| < \frac{1}{10} \min(\pi - \frac{a_x}{K}, \frac{2a_x}{K} - \pi)$  for  $i = 0, \dots, K-1$ . The configuration

$$G = \bigcup_{j \in \mathbf{Z}} \left( \bigcup_{i=0}^{K-1} \tilde{\Gamma} + i \frac{\vec{a}}{K} + \mathbf{d}_i + j\vec{a} \right)$$

yields a self-translating surface invariant under the translation by  $\vec{a}$  if  $\bar{\tau}$  is small enough. Indeed, Theorem 3 states that the smaller  $\bar{\tau}$  is, the closer the constructed self-translating surface is to the initial configuration  $G$ . Therefore, for  $\bar{\tau}$  small enough, the self-translating surface will not be invariant under translations by  $i \frac{\vec{a}}{K}$ ,  $i = 1, \dots, K-1$ .

**Case 3:**  $\pi/2 > a_x > 0$  with  $a_x \neq \pm \frac{\pi}{q}$ ,  $q \in \mathbf{N}$ . In this case, the periodic family of grim reapers is just

$$G = \bigcup_{j \in \mathbf{Z}} \tilde{\Gamma} + j\mathbf{a}.$$

We now show that it is in general position. Let  $K$  be the smallest integer so that  $Ka_x > \pi$ . From the conditions on  $a_x$ ,  $(K-1)a_x \neq \pi$  and the distance between any two asymptotic planes of  $G$  is either  $a_x$ ,  $\pi - (K-1)a_x$ , or  $Ka_x - \pi$ . The last step is to prove that there are no triple intersection by contradiction. In what follows, we consider  $G \cap \{z = 0\}$  to simplify the vocabulary. Without loss of generality thanks to the periodicity, we can assume that the left-most grim reaper is  $\tilde{\Gamma}$  and that for

two integers  $i, j$  with  $0 < i < j < K$ , we have  $\tilde{\Gamma} \cap (\tilde{\Gamma} + i\vec{a}) \cap (\tilde{\Gamma} + j\vec{a}) = \{p\}$ . Using the periodicity of  $G$ , we have that  $(\tilde{\Gamma} + (j-i)\vec{a}) \cap (\tilde{\Gamma} + j\vec{a}) = \{p + (j-i)\vec{a}\}$  and  $(\tilde{\Gamma} + j\vec{a}) \cap (\tilde{\Gamma} + (j+i)\vec{a}) = \{p + j\vec{a}\}$ . The grim reaper  $\tilde{\Gamma} + j\vec{a}$  passes through the three collinear points  $p$ ,  $p + (j-i)\vec{a}$ , and  $p + j\vec{a}$ , which is not possible, therefore there can not be any triple intersection.

### 3. INITIAL CONFIGURATION

In this section, the third dimension does not add any information, so we work with cross-sections in the  $xy$ -plane. We start by giving uniform lower bounds on all the intersection angles.

**Lemma 4.** *Given  $G = \bigcup_{j \in \mathbf{Z}} \bigcup_{n=1}^{N_\Gamma} (\tilde{\Gamma}_n + j\vec{a})$  a family of grim reapers in general position, there exist positive numbers  $\delta$  and  $\delta_\Gamma$  such that*

- *the four angles formed at the intersection of any two grim reapers are greater than  $30\delta_\Gamma$ ,*
- *any tangent vector to a grim reaper at an intersection forms an angle greater than  $30\delta_\Gamma$  with  $\vec{e}_y$ ,*
- *the arc length distance on the grim reapers between any two intersection points of  $G \cap \{z = 0\}$  is greater than  $2\delta$ .*

*Proof.* Let us recall that each  $\tilde{\Gamma}_n$  is the surface  $\tilde{\Gamma}$  from (2) shifted by  $(\tilde{b}_n, \tilde{c}_n, 0)$ . Let  $K$  be the smallest integer for which

$$K|a_x| \geq \left( \pi + \max_{1 \leq n \leq N_\Gamma} \tilde{b}_n - \min_{1 \leq n \leq N_\Gamma} \tilde{b}_n \right).$$

In Lemma 3 in [5], we proved these properties for a finite family of grim reapers in general position. Lemma 4 here follows because  $\bigcup_{1 \leq j \leq K} \bigcup_{n=1}^{N_\Gamma} (\tilde{\Gamma}_n + j\vec{a})$  is a finite family in general position and contains all the relevant intersection points. Note that part (i) of Definition 2 in [5] requires that  $|\tilde{b}_n - \tilde{b}_m - k\pi| > \varepsilon$  for  $n \neq m$  and for all integers  $k$ . This is more restrictive than necessary and can be replaced by (i) Definition 2 given here without any changes in the rest of [5]. Moreover, the angle  $\pi/2 - 30\delta_\Gamma$  should be replaced by  $\pi/2$  in (i) Lemma 3 of [5].  $\square$

**3.1. Construction of a flexible initial configuration.** There are exactly four unit tangent vectors emanating from each intersection point  $p_k$ ,  $k = 1, \dots, N_I$ . We denote them by  $v_{k1}$ ,  $v_{k2}$ ,  $v_{k3}$ , and  $v_{k4}$ , where the number refers to the order in which they appear as we rotate from  $\vec{e}_y$  counterclockwise. The goal of this section is to perturb  $G$  into a configuration  $\bar{G}$  for which the tetrad of directing vectors  $\bar{T}_k = \{\bar{v}_{k1}, \bar{v}_{k2}, \bar{v}_{k3}, \bar{v}_{k4}\}$  satisfies

$$\angle(-\bar{v}_{k1}, \bar{v}_{k3}) = 2\theta_{k,1}, \quad \angle(-\bar{v}_{k2}, \bar{v}_{k4}) = 2\theta_{k,2},$$

where  $\theta_{k,1}$  and  $\theta_{k,2}$  are two small angles. These angles measure how much the pairs of vectors  $(v_{k1}, v_{k3})$  and  $(v_{k2}, v_{k4})$  fail to point in opposite directions. The process of changing the directing vectors at an intersection is called *unbalancing* and will help us deal with small eigenvalues of the linear operator  $L = \Delta + |A|^2$ , which is the operator associated to normal perturbations of the mean curvature  $H$ . As we scale down the desingularizing Scherk surfaces to fit into a small neighborhood of the intersection lines, the contribution of  $H$  dominates and the equation  $Lv = E$  will capture the behavior of the linearized equation associated to (1) on the desingularizing surface. The operator  $L$  has a kernel of dimension 3 generated

by  $\vec{e}_x \cdot \nu$ ,  $\vec{e}_y \cdot \nu$ , and  $\vec{e}_z \cdot \nu$ . We can rule out  $\vec{e}_z \cdot \nu$  by imposing a symmetry, namely that the final surface be invariant under the reflection  $z \mapsto -z$ . To deal with the other two eigenfunctions at each intersection, one can add or subtract a linear combination of eigenfunctions to the inhomogeneous term  $E$  in order to land in the space perpendicular to the kernel, where  $L$  is invertible. The key to a successful construction is to be able to generate these linear combinations by slight unbalancing of the initial configuration.

**Proposition 5.** *Given  $\underline{\theta} := \{\theta_{k,1}, \theta_{k,2}\}_{k=1}^{N_I}$ , one can perturb  $G$  into a configuration  $\bar{G}$  so that the tetrad of directing vectors  $\bar{T}_k = (\bar{v}_{k1}, \bar{v}_{k2}, \bar{v}_{k3}, \bar{v}_{k4})$  at  $\bar{p}_k$ , the  $k$ th intersection point of  $\bar{G}$ , satisfies  $\theta_1(\bar{T}_k) = \theta_{k,1}$  and  $\theta_2(\bar{T}_k) = \theta_{k,2}$  for  $k = 1, \dots, N_I$ . Moreover, there exists a constant  $\delta'_\theta > 0$  depending only on  $N_I$ ,  $\delta_\Gamma$ , and  $\varepsilon$  such that  $\bar{G}$  is embedded if  $|\underline{\theta}| < \delta'_\theta$ .*

Before starting the proof, we give some terminology. We say that two points  $p$  and  $q$  are equivalent and write  $p \sim q$  if there is a  $k \in \mathbf{Z}$  for which  $p + k\vec{a} = q$ . In the proof below, we take the quotient of the configurations with the respect to this equivalence and denote both the configuration and the corresponding quotient by the same letter. The *edges* of a configuration  $G$  are the closures of the bounded connected pieces of  $G \setminus \{\text{intersection points of } G\}$ . The *rays* are the closure of the unbounded pieces. We say two edges (rays, or grim reapers) are equivalent when one is the image of the other under translation by a multiple of  $\vec{a}$ .

In the case of finitely many grim reapers, we started by unbalancing at intersection points on the left side and propagated the perturbations to the right. Because there were finitely many intersection points, we accumulated finitely many error terms, which could then be bounded. Here, the strategy is to fix a “low edge” on each grim reaper then modify the intersection points and pieces of grim reapers as we move upward.

*Proof. Part 1.* We start defining the bottom of the configuration given by

$$(3) \quad \mathcal{B}_1 = \{q \in G \mid y(q) = \min_{q' \in G, x(q')=x(q)} y(q')\}.$$

Because  $G$  is periodic and connected,  $\mathcal{B}_1$  is connected and composed of edges from distinct grim reapers. We can assume by renumbering if necessary, that  $\mathcal{B}_1$  contains edges from the first  $N_1$  grim reapers and also contains the first  $N_1$  intersection points,  $1 \leq N_1 \leq N_\Gamma$ . We define  $\bar{\mathcal{B}}_1 := \mathcal{B}_1$  and take it as part of  $\bar{G}$ . All the intersection points  $\bar{p}_k := p_k$ ,  $k = 1, \dots, N_1$  are now fixed as well as the second and third directing vectors,  $\bar{v}_{k2} := v_{k2}$  and  $\bar{v}_{k3} := v_{k3}$  for  $k = 1, \dots, N_1$ .

For an intersection point  $p \in G$ , we define its *level* (see Figure 1):

- Level 1: an intersection point is at *level* 1 if it is in  $\bar{\mathcal{B}}_1$ .
- Level  $j$ : an intersection point is at *level*  $j$  if it is not in any level below  $j$  and if it is the endpoint of two different edges with level  $(j-1)$  endpoints.
- Level  $\infty$ : an intersection point is at *level*  $\infty$  if it is not at any finite level.

The number of nonempty levels is finite because the number of intersection points is finite. We denote by  $N_{(1,j)}$  the number of intersection points at level  $j$  or below and we can assume by renumbering if necessary that  $p_k$ ,  $1 \leq k \leq N_{(1,j)}$  is of level  $j$  or lower.

*Algorithm for constructing a partial flexible configuration.* We now show how to construct the new level  $j$  intersection points assuming that the points  $\bar{p}_k$ ,  $k =$

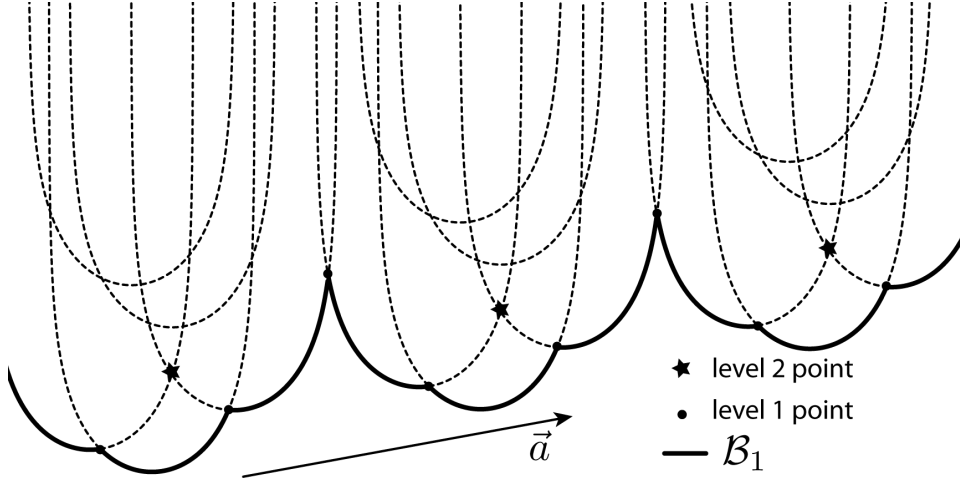


FIGURE 1. Levels of the points of intersection in Part 1. The intersection points not marked are at level  $\infty$ .

$1, \dots, N_{(1,j-1)}$  are defined as well as their directing vectors  $\bar{v}_{k2}$  and  $\bar{v}_{k3}$ . At  $\bar{p}_k$ ,  $k = 1, \dots, N_{(1,j-1)}$ , the directions  $\bar{v}_{k1}$  and  $\bar{v}_{k4}$  are determined by  $\theta_{k,1}$  and  $\theta_{k,2}$ . We attach two pieces of grim reapers at  $\bar{p}_k$ , with tangent directions given respectively by  $\bar{v}_{k3}$  and  $\bar{v}_{k4}$ . The point  $\bar{p}_l$  corresponding to a level  $j$  point  $p_l \in G$  is the intersection of the corresponding newly modified pieces of grim reapers emanating from its neighbor level  $(j-1)$  (or lower) points. The second and third directing vectors at  $\bar{p}_k$  are given by the newly modified grim reaper pieces.

Using the algorithm, we construct new intersection points corresponding to points of  $G$  with finite level. We complete Part 1 by finding the directions  $\bar{v}_{k3}$  and  $\bar{v}_{k4}$  for all the points with highest finite level and attaching modified pieces of grim reapers with tangents in these directions. We denote the resulting partial configuration  $\bar{G}_1$  and note that  $\bar{G}_1$  does not have any information from  $\tilde{\Gamma}_n$ ,  $n > N_1$ .

Let us assume that Part (m-1) has been completed and we obtained the partial configuration  $\bar{G}_{(m-1)}$  involving the grim reapers  $\tilde{\Gamma}_n$ ,  $n \leq N_{m-1} < N_\Gamma$ .

**Part m.** We define  $G_m^* := \bigcup_{j \in \mathbf{Z}} \bigcup_{n > N_{m-1}} \tilde{\Gamma}_n + j\vec{a}$  to be the rest of the configuration and the bottom of  $G_m^*$  by

$$\mathcal{B}_m = \{q \in G_m^* \mid y(q) = \min_{q' \in G, x(q')=x(q)} y(q')\}.$$

Let  $\bar{\mathcal{B}}_m$  be the closure of a bounded connected component of  $\mathcal{B}_m \setminus \bar{G}_{(m-1)}$ . We fix  $\bar{\mathcal{B}}_m$ , therefore the intersection points in the interior of  $\bar{\mathcal{B}}_m$  are fixed as well as two of their directing vectors. For an endpoint of  $\bar{\mathcal{B}}_m$ , the direction pointing toward the interior of  $\bar{\mathcal{B}}_m$  is fixed, and we fix the edge joining the endpoint to an intersection point of  $\bar{G}_{(m-1)}$ . These two directing vectors for the endpoints are not necessarily in the second and third directions but they are not opposite. We can assume by renumbering if necessary, that  $\mathcal{B}_m$  contains edges from the grim reapers  $\tilde{\Gamma}_n$ ,  $N_{m-1} < n \leq N_m$ .

For an intersection point  $p \in G$ , we redefine its *level* (see Figure 2):

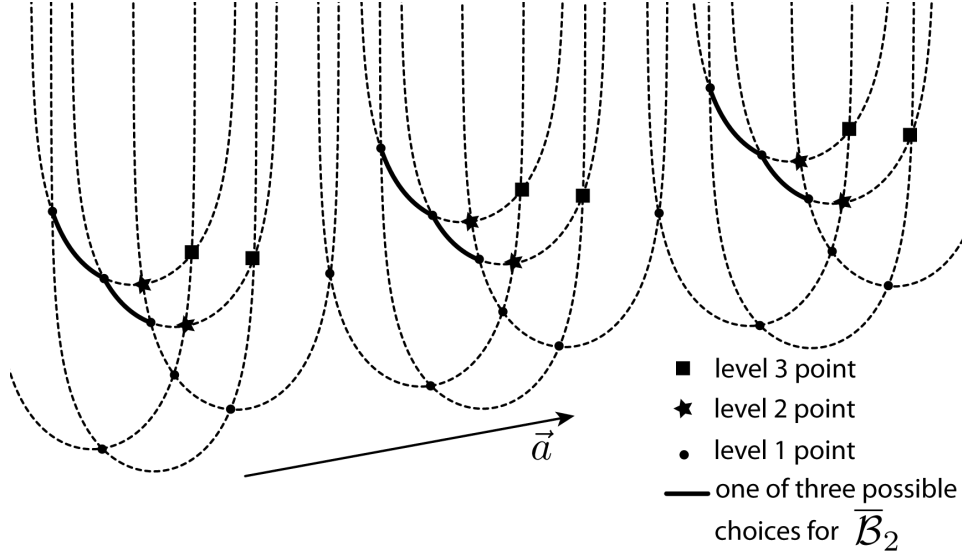


FIGURE 2. Levels of the points of intersection in Part 2 for this choice of  $\overline{\mathcal{B}}_2$ . Not to overcrowd this figure, we took  $\theta_{k,j} = 0$  for  $k = 1, \dots, 4$  and  $j = 1, 2$  so that  $\overline{G}_1 = \bigcup_{j \in \mathbf{Z}} \bigcup_{n=1}^{N_1} \tilde{\Gamma}_n + j\vec{a}$ .

- Level 1: an intersection point  $p$  is at *level 1* if its corresponding constructed point  $\bar{p}$  is in  $\overline{\mathcal{B}}_m$  or in  $\overline{G}_{(m-1)}$ .
- Levels  $j$  and  $\infty$  are defined as in Part 1.

We can use the *algorithm for constructing a partial flexible configuration* for constructing new level  $j$  intersection points assuming that the intersection points corresponding to level  $(j-1)$  or lower are already defined, as well as two of their directing vectors. Although the two fixed directing vectors are not necessarily pointing in the second and third direction in this case, the algorithm is easily modified. We finish Part  $m$  by finding the last two directing vectors for all the points with highest finite level and attaching modified pieces of grim reapers with tangents in these directions. The resulting partial flexible configuration is denoted by  $\overline{G}_m$ .

There are finitely many parts because the number of grim reapers in one period is finite and in each part, we modify finitely many intersection points so the construction ends in finitely many steps. The choice of  $\overline{\mathcal{B}}_m$ ,  $m > 1$  may not be unique, but after the first time we choose all the  $\overline{\mathcal{B}}_m$ , we have an order in which to perturb the intersection points of  $G$ ; we will always keep the same order for the same initial configuration  $G$ . The existence of the constant  $\delta'_\theta$  follows from Proposition 5 [5], which proves the smooth dependence between the position of the intersection points, grim reaper positions, and directing vectors.  $\square$

#### 4. INITIAL SURFACES AND THE PERTURBATION PROBLEM

The rest of construction follows along the steps of [5] and we outline them here.

- We construct desingularizing surfaces to replace intersection lines by appropriately dislocating and bending original Scherk minimal surfaces. The dislocation is to ensure that the desingularizing surface can fit into an

unbalanced configuration and to deal with small eigenvalues of the linear operator. The main purpose of the bending is to guarantee exponential decay of solutions to the linearized equation on this piece of surface.

- We define a family of initial surfaces, which are approximate solutions to (1), by fitting appropriately bent and scaled Scherk surfaces at the intersection lines. The fitting of the desingularizing Scherk surfaces moves the edges and the grim reapers ends slightly; however, it does not change the position of the intersection lines so the periodicity can be preserved. The errors created can be controlled (see Proposition 33 and Corollary 34 [5]).
- We study the linear operator  $\mathcal{L}$  associated to normal perturbations of  $H - \vec{e}_y \cdot \nu$ . The linearized equation  $\mathcal{L}v = E$  can be solved, modulo the addition of correction functions to the inhomogeneous term. These correction functions can be generated by the flexibility build into  $\overline{G}$  and the construction of the desingularizing surfaces (see Propositions 38 and 45 [5]).
- With the correct norms on the initial surfaces, we apply a fixed point theorem to find exact solutions (see Theorem 48 [5]). The number of parameters introduced during the construction of the initial approximate solutions is finite because the number of intersections within a period is finite. At this point, it is worth noting that the obstacle that prevents us from constructing a infinite dimensional family of self-translating surfaces based on an initial configuration such as

$$\bigcup_{j \in \mathbf{Z}} \tilde{\Gamma} + j\vec{a} + \mathbf{b}_j,$$

where  $\mathbf{b}_j$  are vectors in  $\mathbf{R}^2 \times \{0\}$  with  $|\mathbf{b}_j| < \delta$ , is not the lack of flexibility but the fact that we would have infinitely many parameters. Even if the parameters are uniformly bounded, the current proof, which uses a fixed point theorem in the final step, would not apply here because an infinite countable product of compact intervals is not a compact Banach space.

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